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Dirac's Superposition of Pure States Extended to the Statistical Operators

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A definition of superposition relation for the statistical operators is proposed which is equivalent to the one of Varadarajan. It is found that the superposition relation is preserved under a general linear dynamics and under tensor product.

1. THE SUPERPOSITION RELATION

It is well known that the pure states of a purely quantum (e.g., without superselection rules) physical system Σ with separable complex Hilbert space H are represented in the standard formulation of quantum mechanics by the rays $[\psi]$ $(\psi \in H)$ of H (Dirac, 1947). In that context the superposition relation for the pure states is expressed by the fact that the normalized to one linear combination of representative vectors $\psi_k \in [\psi_k]$ $(k \in I,\ I$ countable), namely, the relation

$$\psi = \Sigma_k c_k \psi_k \tag{1.1}$$

produces new pure states which are different, in general, from the original ones.

This assumption has, in particular, the consequence that the dynamics which the physical system undergoes is provided by the Schrödinger equation (Dirac, 1947), which, owing to its linearity, preserves the superposition relation for the pure states.

In quantum statistical mechanics instead the functions that are used to represent the states of the physical system are the so-called *statistical*

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operators (or density operators) on H, namely, the positive trace class operators on H with trace one $K(H)$. By decomposing a statistical operator $\rho \in K(H)$ in terms of one-dimensional projections (repeating the eigenvalues if necessary)

$$\rho = \Sigma_i \gamma_i P_{\psi_i} \qquad (\|\psi_i\| = 1) \tag{1.2}$$

the physical interpretation of ρ is that it represent the state of Σ as a statistical mixture of the pure states $[\psi_i]$ with relative frequency γ_i ($i = 1,2,3,\ldots$).

One could then ask whether it is possible to give a notion of superposition directly for the statistical operators.

In this note we give a positive answer to that question in the sense that we propose an extension of Dirac's superposition of pure states to the statistical operators.

More precisely, we say that a statistical operator $\rho \in K(H)$ is a superposition of the family of statistical operators $\{\rho^\alpha\} \subset K(H)$ if, once the ρ^α's have been decomposed in terms of one-dimensional projections

$$\rho^\alpha = \Sigma_k \gamma_k^\alpha P_{\psi_k^\alpha}$$

it comes out that every pure state $[\psi_i]$ obtained from ρ as in (1.2) is a Dirac superposition of pure states of the family $\{[\psi_k^\alpha]\}$. It is immediate that the given definition of superposition is equivalent to the assumption that ρ is a superposition of the statistical operators $\{\rho^\alpha\}$ if and only if

$$[\rho] \leqslant \bigvee_\alpha [\rho^\alpha] \tag{1.3}$$

where $[\rho]$ denotes the range of ρ as an operator in H; $\bigvee_\alpha [\rho^\alpha]$ stands for the closure of the linear span generated by the subspaces $[\rho^\alpha]$ of H and $\leqslant$ stands for set inclusion.

Before testing the validity of our definition of superposition in the case of the dynamics and in the coupling of physical systems we want to show that, as a matter of fact, our definition is equivalent to another definition of superposition which has been proposed for the first time (as far as the author knows) by Varadarajan (Varadarajan, 1968) in the context of the logic approach to quantum mechanics.

According to this approach, when applied to the Hilbert model, the closed subspaces $L(H)$ of H are interpreted as representing the lattice of equivalent classes of yes–no experiments (experimental propositions or simply propositions) on Σ. The states are represented in that context by the

σ-additive measures with total mass 1 $S(H)$ on $L(H)$, namely, by the functions $s: L(H) \to [0, 1]$ with

$$\begin{array}{ll} \text{(i)} & s(H) = 1 \\ \text{(ii)} & s(\bigvee_i x_i) = \Sigma_i s(x_i) \end{array} \tag{1.4}$$

where $\{x_i\}$ is any countable family of mutually orthogonal subspaces of H.

According to Gleason's theorem (Gleason, 1957) for every $s \in S(H)$ there is one and only one statistical operator $\rho \in K(H)$ such that

$$s(x) = s_\rho(x) = \operatorname{Tr} \rho P^x \tag{1.5}$$

P^x being the orthogonal projection on H with range x. When applied to the Hilbert model, the definition of superposition given by Varadarajan is as follows. The state $\rho \in S(H)$ is said to be a superposition of the states $\{s_\rho \alpha\} \subset S(H)$ if

$$\operatorname{Tr} \rho^\alpha P^x = 0 \forall \alpha \Rightarrow \operatorname{Tr} \rho P^x = 0, \qquad x \in L(H) \tag{1.6}$$

or equivalently

$$\operatorname{Tr} \rho^\alpha P^y = 1 \forall \alpha \Rightarrow \operatorname{Tr} \rho P^y = 1, \qquad y \in L(H) \tag{1.7}$$

The last relation (1.7) can be equivalently written as

$$L(\rho) \supset L(\{\rho^\alpha\}) \tag{1.8}$$

where for every $D \subset K(H)$ it has been defined

$$L(D) = \{x \in L(H): \operatorname{Tr} \sigma P^x = 1 \forall \sigma \in D\} \equiv \bigcap_{\sigma \in D} L(\sigma) \tag{1.9}$$

By using the spectral decomposition of a density operator, it is not difficult to show that the following formula holds:

$$L(D) = \{x \in L(H): P^x \rho = \rho\} \tag{1.10}$$

From this we have the further properties

$$x \in L(D), y \in L(H) \Rightarrow x \vee y \in L(D)$$

$$x, y \in L(D) \Rightarrow x \wedge y \in L(D)$$

$$\wedge L(D) \equiv \bigwedge_{x \in L(D)} x \in L(D)$$

that is, by using the language of the lattice theory, $L(D)$ is a dual principal ideal (Birkhoff, 1967) of the lattice $L(H)$, so that it can be equivalently represented as the set of propositions

$$L(D)=\{x\in L(H)\colon \wedge L(D)\leqslant x\} \tag{1.11}$$

Moreover there also holds (Gorini and Zecca, 1975; Berzi and Zecca, 1974) the formula

$$\wedge L(D)=\bigvee_{\sigma\in D}[\sigma] \tag{1.12}$$

By taking into account the last considerations we have then that the relation (1.8) holds if and only if the following holds:

$$\wedge L(\rho)=[\rho]\leqslant\wedge L(\{\rho^\alpha\})=\bigvee_\alpha[\rho^\alpha]$$

which is nothing other than our definition (1.3) of superposition of statistical operators. The proof that our definition of superposition coincides with the one of Varadarajan is thus completed.

2. SUPERPOSITION, DYNAMICS, AND TENSOR PRODUCT

We want now to show that as the superposition of pure states is preserved under a linear evolution of the physical system also here the superposition relation of the statistical operators is preserved under the most general linear (possibly irreversible) dynamical evolution to which the physical system is subjected. To do this we need a definition.

Definition. A *dynamical map* B for the physical system Σ is a map from the density operators $K(H)$ into themselves which is affine, namely, such that

$$B(\alpha\rho+(1-\alpha)\sigma)=\alpha B\rho+(1-\alpha)B\sigma,\qquad \rho,\sigma\in K(H),\alpha\in[0,1]$$

(B is not assumed to be onto nor one to one).

In Zecca (1980) the following result has been shown.

Proposition. Any dynamical map B of the physical system has the property

$$L(\rho)\supset L(D)\Rightarrow L(B\rho)\supset L(BD),\qquad \rho\in K(H),\, D\subset K(H)$$

From the equivalence shown above between the relations (1.3) and (1.8) we have thus reached our goal. Indeed a dynamical evolution for the physical system for which a statistical interpretation is allowed is described in general in terms of a one-parameter family $t \to B_t$ of dynamical maps with the interpretation that if ρ is the state of the system at time $t=0$ then $B_t\rho$ represents the state of the system at time t.

A general example of a one-parameter family of dynamical maps is the one given by the motion of the physical system governed by a homogeneous generalized master equation, which gives a formally exact description of the time evolution of a quantum open system coupled to its surroundings (Haake, 1960; Lanz, Lugiato, and Ramella, 1971).

Since B_t has not been assumed to be, in general, a bijection, the superposition of the statistical operators is preserved also under an irreversible dynamical evolution. For instance, $t \to B_t$ $(t \geqslant 0)$ could be the one-parameter semigroup of dynamical maps obtained from the solution of a Markovian master equation. These equations are widely used in the phenomenological treatment of open systems and are also a useful tool to approximate in the weak coupling limit (Davies, 1974) or in the singular reservoir limit (Hepp and Lieb, 1973) the generalized master equation (see also Frigerio, Gorini, Kossakowski, Sudarshan, and Verri, 1978 and references therein).

A special case of the above-mentioned family $t \to B_t$ could be the weakly continuous one-parameter group of unitary automorphisms of $K(H)$ given by the motion of a strictly isolated physical system, namely,

$$B_t\rho = U_t\rho U_t^+$$

where $t \to U_t$ is a weakly continuous one-parameter group of unitary operators on H, whose generator, which exists by Stone's theorem (Reed and Simon, 1972), represents the Hamiltonian of the system. We remark here that a dynamical evolution for the physical system which preserves the superposition relation for the statistical operators need not to be a linear evolution, as has been pointed out by the mathematical nonlinear example in Zecca (1976).

Finally we want to check that our superposition relation for the statistical operators is compatible also with the coupling of physical systems, namely, it is preserved under tensor product. To see this let $\tilde{\Sigma}$ be a second irreducible quantum physical system with separable Hilbert space $\tilde{H}$. Let then $\rho \in K(H)$ be a superposition of the statistical operators $D \subset K(H)$ for Σ and $\tilde{\rho} \in K(\tilde{H})$ be a superposition of the statistical operators $\tilde{D} \subset K(\tilde{H})$ for $\tilde{\Sigma}$. We have then equivalently, according to our original definition of superposition of statistical operators, the assumptions

$$[\rho] \leqslant \bigvee_{\sigma \in D} [\sigma] \qquad \text{and} \qquad [\tilde{\rho}] \leqslant \bigvee_{\tilde{\sigma} \in \tilde{D}} [\tilde{\sigma}]$$

By using the property of the tensor product function there follows

$$[\rho]\otimes[\tilde{\rho}]=[\rho\otimes\tilde{\rho}]\leqslant \bigvee_{\substack{\sigma\in D\\ \tilde{\sigma}\in\tilde{D}}}[\sigma]\otimes[\tilde{\sigma}]=\bigvee_{\substack{\sigma\in D\\ \tilde{\sigma}\in\tilde{D}}}[\sigma\otimes\tilde{\sigma}]$$

which again by our definition of superposition means that $\rho\otimes\tilde{\rho}$ is a superposition of the set of states $D\otimes\tilde{D}=\{\sigma\otimes\tilde{\sigma}\colon \sigma\in D, \tilde{\sigma}\in\tilde{D}\}$, that is the superposition relation is preserved under tensor product.

We have shown that the superposition relation for statistical operators is preserved under any (linear) dynamics of the physical system and under the coupling of physical systems. These reasons should be sufficient to make the definition of superposition a good definition.

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